MMAT5030 Notes 2

1 Uniform Convergence of Series of Functions

Let $f_n, n \geq 1$, be functions defined on some interval I. We consider the series of functions $\sum_{n=1}^{\infty} f_n$. This series **pointwisely converges** to a function f if for each $x \in I$, the series of numbers $\sum_{n=1}^{\infty} f_n(x)$ converges to the number $f(x)$. In other words, for each $x \in I$ and $\varepsilon > 0$, there is some N_0 depending on x and ε such that

$$|\sum_{n=1}^{N} f_n(x) - f(x)| < \varepsilon \ , \quad \forall N \geq N_0.$$

It **uniformly converges** to f if the number N_0 can be chosen independent of x, that is, for $\varepsilon > 0$, there is some N_0 such that

$$|\sum_{n=1}^{N} f_n(x) - f(x)| < \varepsilon \ , \quad \forall N \geq N_0, \quad \forall x \in I \ .$$

It is clear that uniform convergence implies pointwise convergence but the converse is not true. Uniform convergence has many nice properties. We list three of them.

Theorem 1 (Continuity Theorem). Suppose that each f_n is continuous on I and the series $\sum_{n=1}^{\infty} f_n$ uniformly converges to f. Then f is continuous on I.

In brief, uniform convergence preserves continuity. Here is an example of pointwise but not uniformly convergent series. It does not preserve continuity.

Example 1. Recall the function $f(x) = x, x \in (-\pi, \pi]$, extended as a 2π-periodic function, is piecewise smooth with jumps at $(2n+1)\pi$. Its Fourier series is given by

$$2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \ ,$$

(see pg 26 in Text). Now consider f as defined on $[0, 2\pi]$. It is smooth except jumps at π. According to the main convergence theorem (Theorem 2.1 in Text), for $x \in [0, 2\pi], x \neq \pi$, the series converges to $f(x)$. At $x = \pi$, it converges to 0. Hence this series converges pointwisely on $[0, 2\pi]$. However, it cannot be uniformly convergent. For, if it is, by Continuity Theorem, f must be continuous on $[0, 2\pi]$ which is not true.

Theorem 3 (Integration Theorem). Suppose that $f = \sum_{n=1}^{\infty} f_n$ is uniformly convergent where f_n's are piecewise continuous on $[a, b]$. The series $\sum_{n=1}^{\infty} F_n$, where $F_n(x) = \int_a^x f_n(t)\, dt$, converges uniformly to $F(x) = \int_a^x f(t) dt$.

In this theorem the base point a in the definition of the primitive functions can be replaced by any other point $x_0 \in [a, b]$. The following is an application of this theorem. It shows that the Fourier series of any uniformly convergent trigonometric series is equal to itself.

Proposition 4. Consider a pointwise convergent trigonometric series

$$\frac{\alpha_0}{2} + \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx) \ , \quad x \in [-\pi, \pi],$$

and denote it by $f(x)$. In case the convergence is uniform, the Fourier series of f is equal to the series itself.

Proof As the convergence is uniform, f is continuous on $[-\pi, \pi]$ by Continuity Theorem. It is easy to see that the series

$$\frac{\alpha_0}{2} \cos mx + \sum_{n=1}^{\infty} (\alpha_n \cos nx + \beta_n \sin nx) \cos mx \ , \quad m \geq 1,$$

obtained by multiplying $\cos mx$ to both sides of the series, is again uniformly convergent (to $f(x) \cos mx$). By Integration Theorem,

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos mx dx &= \int_{-\pi}^{\pi} \frac{\alpha_0}{2} \cos mx dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} \left(\alpha_n \cos nx \cos mx dx + \int_{-\pi}^{\pi} \beta_n \sin nx \cos mx dx \right) \\ &= \pi \alpha_m \ , \end{aligned}$$

but the left hand side is equal to πa_m, the Fourier coefficient of f. We conclude that $\alpha_m = a_m, m \geq 0$. Similarly, we can verify the other cases.

Theorem 5 (Differentiation Theorem). Suppose that (a) each f_n is continuous on I and the series $\sum_{n=1}^{\infty} f_n$ uniformly converges to f, and (b) each f_n is differentiable on I and $\sum_{n=1}^{\infty} f_n'$ uniformly converges to g. Then f is differentiable and $f' = g$ on I.

Very often we use the notation $\sum_{n=1}^{\infty} f_n$ to denote the pointwise/uniform limit of the series $\sum_{n=1}^{\infty} f_n$. Thus, $\sum_{n=1}^{\infty} f_n$ has two meanings, first it is the notation for a series of functions. Second, it stands for the limit or sum of the series. Using the second meaning, we can express Differentiation Theorem as:

$$\left(\sum_{n=1}^{\infty} f_n(x) \right)' = \sum_{n=1}^{\infty} f_n'(x) \ ,$$

that is, summation and differentiation are commutative. On the other hand, the conclusion of Integration Theorem can be expressed as

$$\int_a^x \left(\sum_{n=1}^{\infty} f_n(t) \, dt \right) = \sum_{n=1}^{\infty} \int_a^x f_n(t) \, dt \ .$$

Given a series of functions, how can we show that it is uniformly convergent? The most common method is Weierstrass' M-Test.

Theorem 6 (M-Test). Let $\sum_{n=1}^{\infty} f_n$ be a series of functions defined on I. Suppose that there exists $a_n, n \geq 1$, satisfying (a) $|f_n(x)| \leq a_n$, for all n and $x \in I$, and (b) $\sum_{n=1}^{\infty} a_n < \infty$. Then $\sum_{n=1}^{\infty} f_n$ is uniformly convergent.

Example 2. Consider the cosine series $\sum_{n=1}^{\infty} \dfrac{\cos nx}{n^2}$. Using $|\cos\theta| \le 1$, we see that $|\cos nx/n^2| \le 1/n^2$. As $\sum_{n=1}^{\infty} 1/n^2 < \infty$, we conclude that the series $\sum_{n=1}^{\infty} \dfrac{\cos nx}{n^2}$ is uniformly convergent by M-Test. Furthermore, since each $\cos nx/n^2$ is continuous, $\sum_{n=1}^{\infty} \dfrac{\cos nx}{n^2}$ is a continuous function by Continuity Theorem.

Example 3. Consider the sine series $\sum_{n=1}^{\infty} \dfrac{\sin nx}{n^n}$. The function $f_n(x) = \sin nx/n^n$ satisfies $f'(x) = \cos nx/n^{n-1}$, $f''(x) = -\sin nx/n^{n-2}$, $\cdots$. Clearly, $|f^{(k)}(x)| \le 1/n^{n-k}$. Using $\sum_{n=1}^{\infty} 1/n^{n-k}$ is convergent for all k, we conclude that the series $\sum_{n=1}^{\infty} f^{(k)}$ is uniformly convergent for all k. A repeated application of Differentiation Theorem shows that $\sum_{n=1}^{\infty} \dfrac{\sin nx}{n^n}$ is an infinitely many times differentiable function.

We apply these results to Fourier series. A series of numbers $\{a_n\}$ is called **rapidly decreasing** if for each k, there is some constant C such that $|a_n| \le C/n^k$ for all n.

Theorem 7. A continuous, piecewise smooth, 2π-periodic function is infinitely many times differentiable if and only if its Fourier coefficients are rapidly decreasing.

Proof. Let a_n, b_n be the Fourier coefficients of f. When f is infinitely many times differentiable, the Fourier coefficients of $f^{(k)}$ tends to 0 as $n \to \infty$ as a consequence of Bessel's Inequality applied to the function $f^{(k)}$. From the relations among the Fourier coefficients of a function and its derivatives (see exercise), it implies

$$n^k|a_n|,\ n^k|b_n| \to 0, \quad n \to \infty.$$

In particular, it means that there is some C such that

$$n^k|a_n|, \quad n^k|b_n| \le C \ ,$$

for each k. Hence a_n, b_n are rapidly decreasing.

Conversely, when the coefficients are rapidly decreasing, $|a_n|, |b_n| \le C/n^k$ for all k. Taking $k = 3$, it implies that the series $\sum(nb_n \sin nx - na_n \sin nx)$ which is obtained from differentiating the Fourier series of f term by term, is uniformly convergent. Since by Theorem 2.5 in Text, the Fourier series of f converges to f uniformly. We can now apply Differentiation Theorem to conclude that f is differentiable and

$$f'(x) = \sum(nb_n \sin nx - na_n \sin nx) \ ,$$

where the convergence is uniform. Repeating this argument, one can show that f is infinitely many times differentiable.

For a detailed discussion on uniform convergence one is referred to chapter 8, Bartle-Sherbert.